The Pumping Lemma for Context-Free Languages

Example. Consider the non-Chomsky grammar

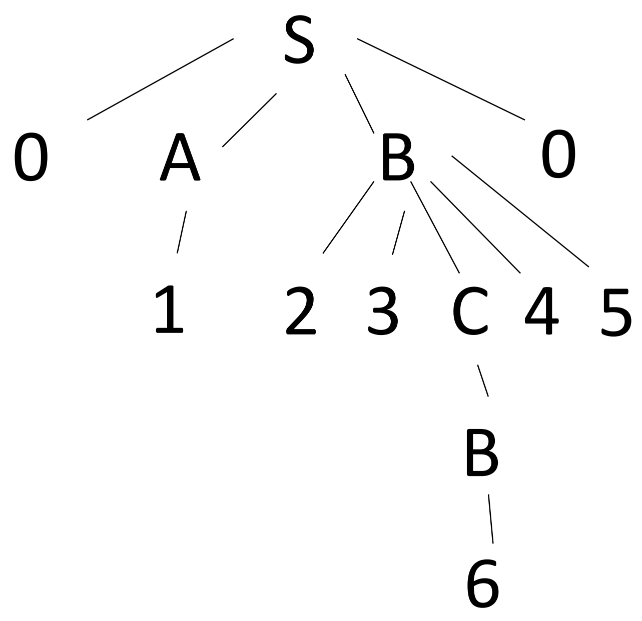
S => 0AB0

A => 1

B => 23C45 | 6

C => B

Parse 01236450



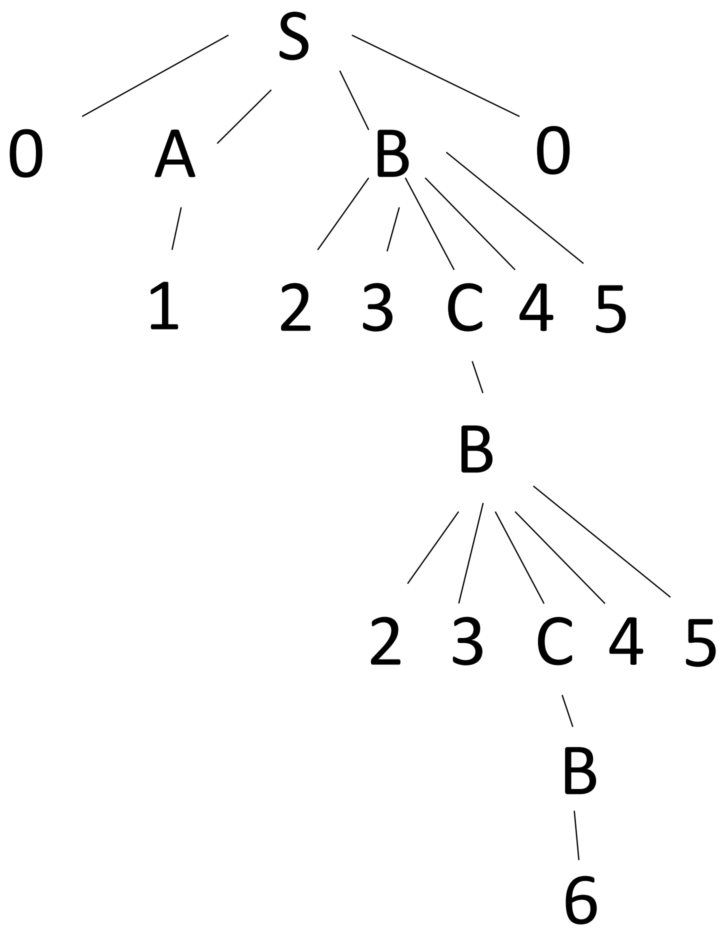
Because of the repeated use of B we could also derive 12323645450

In fact, we could derive

$$01(23)^n6(45)^n0$$

for any n>=0.

This looks something like the Pumping Lemma for regular languages.



We need a fact about binary trees that you have seen before:

Lemma: A binary tree with height n has at most 2^n leaves.
Proof: Simple induction on n.

So a binary tree with more than 2^n leaves must have height greater than n, which means that it must have a path with more than n non-leaf nodes.

The Pumping Lemma for Context-Free Languages (1961 Bar-Hillel, Perles, Shamir): Let $\mathcal{L}$ be a context-free language. Then there is a constant p so that if z is a string in $\mathcal{L}$ and |z|>=p there must be a decomposition z=uvwxy such that

1) |vwx| <= p
2) vx $\neq \varepsilon$ (i.e., both v and x aren't ε)
3) For each i>=0 uv^iwx^iy is in $\mathcal{L}$

Proof of the Pumping Lemma:

Let G be a CNF grammar for $\mathcal{L}$-{ε}. Let N be the number of nonterminal symbols of G. Let p = 2^N+1. Let z be a string in $\mathcal{L}$ with |z| >= p. Since the parse tree for z must have more than 2^N leaves, it must have height greater than N and so it must have a path with more than N nonterminal symbols. This path must have a repeated nonterminal; call this A.

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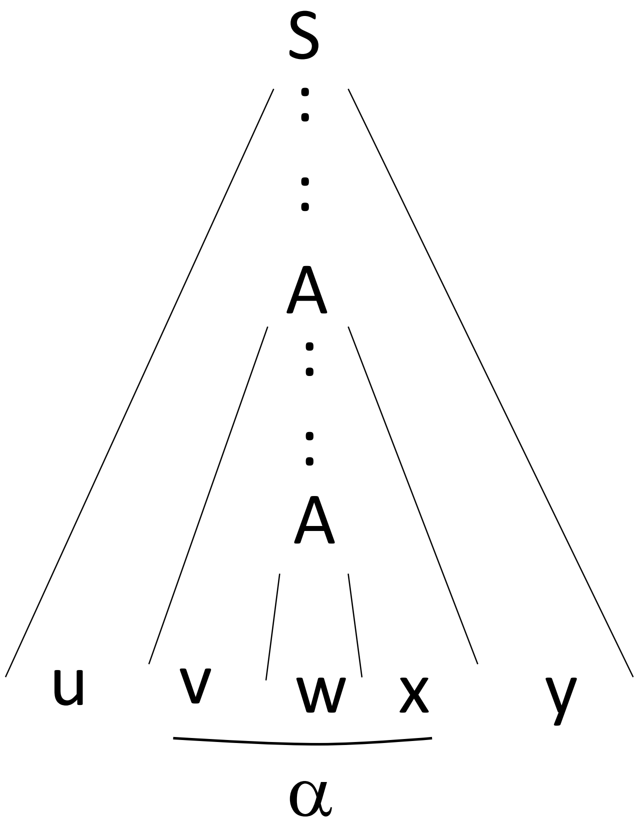

Let α be the portion of z derived from the upper A symbol. Let u be the portion of z that precedes α, y the portion that follows α. Similarly, let w be the portion of α derived from the lower A, v and x the portions of α that precede and follow w. So z=uvwxy.

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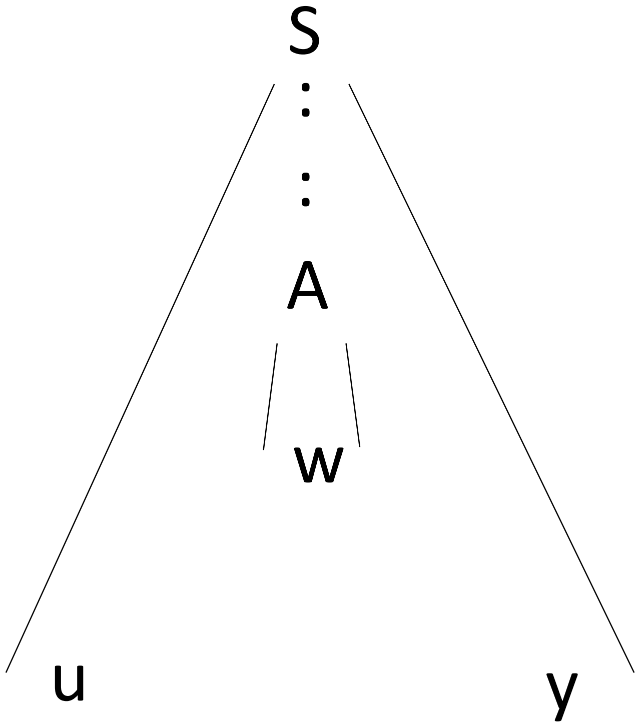
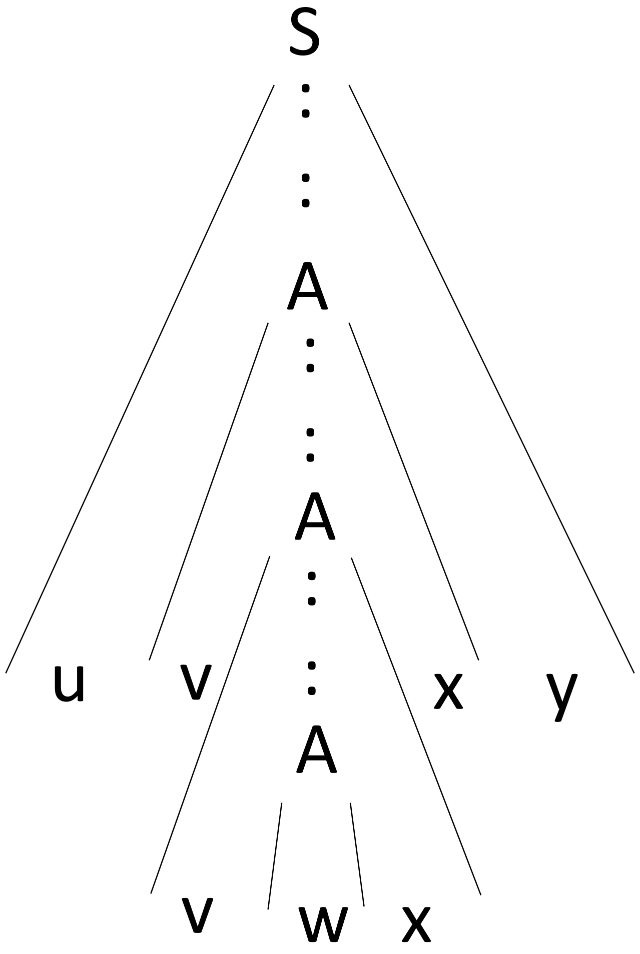
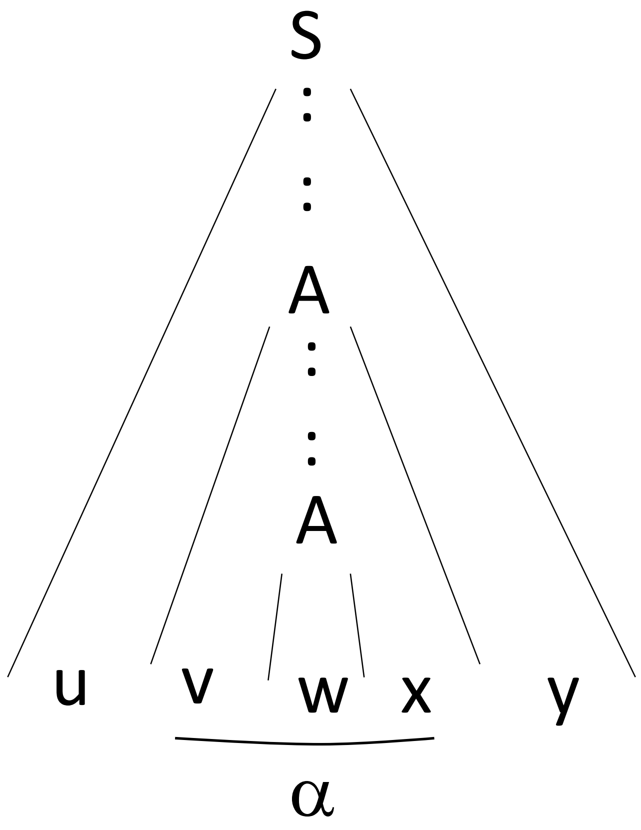
We can pump 0 times by replacing the upper A with the lower one:



We can pump twice by replacing the lower A with a copy of the upper A:



And so forth. We can pump n times for every n >= 0.



If we choose the upper A to be the lowest repeated nonterminal in the original parse tree for z, then the subtree rooted at A has height n or less so it has no more than 2^N leaves. So $|\alpha| <= 2^N < p$.

Since we can't have $A \stackrel{*}{\Rightarrow} A$, $vx \neq \varepsilon$.

This completes the proof.

Example: Show $\{1^n2^n3^n \mid n >= 0\}$ is not context-free.

Suppose it is context-free; let p be its pumping constant. Let $z = 1^p2^p3^p$. Consider any decomposition z=uvwxy where $|vwx| <= p$. Then vwx contains at most 2 of the digits {1,2,3}, so uv^2wx^2y cannot have the same numbers of all 3 digits. So z can't be pumped, contradicting the Pumping Lemma.

Example: Show that $\{ww \mid w \in (0+1)^*\}$ is not context-free

Suppose it is context-free. Let p be its pumping constant. Let $z=0^p1^p0^p1^p$. Consider any decomposition z=uvwxy where $|vwx| <= p$. vwx can't contain both 0's in the first half and 0's in the second, and it can't contain both 1's in the first half and 1's in the second half. Either way, uv^2wx^2y must have different numbers of 0's or different numbers of 1's between the two halves. So uv^2wx^2y is not in the language and z can't be pumped, contradicting the pumping lemma.

Example: Show that $\{0^{n^2} \mid n \geq 0\} = \{\varepsilon, 0, 0^4, 0^9, 0^{16}, \ldots\}$ is not context-free.

Suppose it is context-free; let p be its pumping constant. Let z = 0^{p^2}
Consider any decomposition z=uvwxy where |vwx| <= p. Then $|uv^2wx^2y| <= |z| + p <= p2+p < (p+1)^2$. So

$p2 <= | uv^2wx^2y | < (p+1)2$, so uv^2wx^2y is not in the language.

This means z is not pumpable, contradicting the pumping lemma.